

YAMABE SPECTRA

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Yamabe Spectra

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0. INTRODUCTION

The Uniformization Program of William Thurston prescribes, for a large class of compact closed three-manifolds, the existence of a hyperbolic metric, i.e. a metric of constant negative curvature - 1. One approach to find such a metric is a two-step variational problem, which we now describe. Consider a compact Riemannian manifold (M^n, g) , and for any metric \tilde{g} in the conformal class $C = C(g)$, define

$$(1) \quad Y(\tilde{g}) = \frac{\int_M R_{\tilde{g}} d\text{Vol}_{\tilde{g}}}{[\text{Vol}(\tilde{g})]^{\frac{n-2}{n}}}$$

Then one easily shows that the critical points \tilde{g} of $Y(\tilde{g})$ in C are metrics of constant scalar curvature [16]. The minimum $\inf Y(\tilde{g})$, $\tilde{g} \in C$ is called the Yamabe invariant of (M, g) and denoted $\lambda(g)$. The solution to the Yamabe problem, obtained in the last thirty years due to the efforts of Yamabe [16], Trudinger [14], Aubin [1], and Schoen [11], [12] establishes the existence of the minimizing metric \tilde{g} in C . For M three-dimensional one knows, under certain topological restrictions, that the constant scalar curvature of a metric on M may not be positive. In fact, hyperbolizable manifolds never carry a metric of nonnegative scalar curvature, as shown by Schoen-Yau [13] and Gromov-Lawson [3]. So for such manifolds, $\lambda(g)$ is always negative. The importance of the functional $C \rightarrow \lambda(g)$ stems from the hope to find a conformal class C , maximizing this functional. Then an easy computation show that the corresponding constant scalar curvature metric \tilde{g} in C is actually Einstein, hence hyperbolic, since $\dim M = 3$. It is more convenient to introduce a modified function $Vol : C \mapsto Vol(C)$, as follows: take the appropriately scaled metric \tilde{g} in C of constant scalar curvature -1 , and denote $Vol(C) = \text{Vol}(\tilde{g})$. We will call the set $\{Vol(C)\}$ the Yamabe spectrum of M .

The expectation for the global minimum of Vol at the hyperbolic metric is justified by the following known facts: firstly, if g_0 is a (necessarily unique up to a diffeomorphism) hyperbolic metric on M , then $(D^2Vol)_{g_0} > 0$, see [4], theorem 8.2. Secondly, if $\dim M = 4$ and g_0 is hyperbolic, then the Euler characteristic and signature computations of Johnson and Millson [4], theorem 8.3 show that Vol attains its global minimum at g_0 .

In the three-dimensional case we do not have Gauss-Bonnet type formulas, and one should look for other ways for estimation Vol . In the present paper, we deal with Haken three-manifolds with infinite first homology group. For every conformal class, C , we introduce an invariant of C , coming from the L^3 -geometry of the Jacobian variety $J^1(M) = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$. We prove that this invariant estimates $Vol(C)$ from below and use it to show our main result:

THEOREM 2. *Let M be a compact oriented homologically atoroidal three-manifold with $\pi_2(M) = 0$ and $H_1(M, \mathbb{R}) \neq 0$. The $\sup_C Vol(C) = \infty$.*

We use our estimates for demonstrating new global obstructions for the Nash isometrical immersions $M^3 \rightarrow N$ of arbitrary codimension, which induce nontrivial map in the first homology. Finally, we establish, for all three-manifolds with the pinched negative curvature $-K \leq K(M) \leq -k < 0$, the following principle: the Thurston genus norm in $H_2(M)$ is uniformly equivalent to the area norm.

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1. THE HOMOLOGICAL PLATEAU PROBLEM

The following version of the existence theorem for minimal surfaces of Sacks-Uhlenbeck-Schoen-Yau (c.f. [10], [13]) has been established recently by Marina Ville [16]:

THEOREM A. *Let M be a compact Riemannian manifold and let $0 \neq z \in H_2(M, \mathbb{Z})$. Then there exists a (branched) minimal immersion f of a compact oriented, possibly disconnected Riemannian surface Σ^g to M , representing z . Moreover, one can take Σ^g of the least possible genus among all surface, representing z and the map f to be globally area minimizing in its homology class.*

For the reader's convenience we sketch the proof here. Start with the following lemma.

LEMMA 1. (comp. [8], [17]). *Let X be a CW-complex and let $z \in H_2(X, \mathbb{Z})/\overline{\pi_2(X)}$. There exists a collection of oriented Riemannian surfaces $\Sigma^{g_1}, \dots, \Sigma^{g_k}$, $g_i \geq 1$ and maps $f_i : \Sigma^{g_i} \rightarrow X$ such that*

- (i) $\sum (2g_i - 2) = \|z\|_g$ (see the section 2).
- (ii) $\sum_i [f_i] = z$
- (iii) For any essential simple loop γ in Σ^{g_i} , $f_{i*}([\gamma_i]) \neq 0$ in $\pi_1(x)$.

PROOF: Start with any collection of Σ^{g_i} satisfying (i) and (ii). Suppose (iii) is not valid, that is, for some γ in Σ^{g_i} , there exists a map $\varphi : D^2 \rightarrow X$ with $\varphi|_{\partial D^2} = f|_{\gamma}$ (we identify γ with ∂D^2). Cut Σ^{g_i} along γ and paste two copies of D^2 along the new boundaries. Denote $\tilde{\Sigma}_i$ the resulting surface and let $\tilde{f}_i : \tilde{\Sigma}_i \rightarrow X$ be the map, obtained by patching φ and f_i . Consider the two following cases.

- 1) γ is not separating. Then genus $(\tilde{\Sigma}_i) = g_i - 1$
- 2) γ is separating. Then if $\tilde{\Sigma}_i^{(1)}$ and $\tilde{\Sigma}_i^{(2)}$ are the two components of $\Sigma^{g_i} - \gamma$, then $\tilde{\Sigma}_i = \tilde{\Sigma}_i^{(1)} \sqcup \tilde{\Sigma}_i^{(2)}$ and genus $(\tilde{\Sigma}^{(1)}) + \text{genus}(\tilde{\Sigma}^{(2)}) = g_i$.

In any case, we have, first, that the new collection of surfaces and maps still represents z , and, second, that $\sum |\chi(\Sigma^{g_i})|$ strictly decreases. This contradicts (i) and thus (iii) is valid.

Returning to the proof of Theorem A, we first note, that by the theorem of Sacks and Ulenbeck, there exists a set of minimal spheres in M , generating $\pi_2(M)$ as a $\pi_1(M)$ module, so their images in $\overline{\pi_2(M)}$ generate $\overline{\pi_2(M)}$. Hence we may work modulo $\overline{\pi_2(M)}$.

Fix $z \in H_2(M)/\overline{\pi_2(M)}$ and consider a collection of surfaces and maps Σ^{g_i} , f_i as in the lemma 1, so that $\sum(2g_i - 2)$ is the least possible. By the Hopf exact sequence, we have $H_2(M)/\overline{\pi_2(M)} = H_2(\pi_1(M))$. Now, the condition (iii) implies, by the theorem of Schoen-Yau [13] and Sacks-Uhlenbeck [10], that there exist, for any i , a branched minimal immersion $\psi_i : \Sigma^{g_i} \rightarrow M$, inducing the same action on $\pi_1(\Sigma^{g_i})$ as f_i , up to a conjugation. In particular, $\Sigma[\psi_i] = z$ in $H_2(\pi_1(M))$, as desired.

We refer to [8] for the further refinement of this result and numerous algebraic applications.

If the dimension of M is three, we may (and will) consider f to be unbranched, by the same argument as in [13].

2. THE THURSTON NORM AND THE AREA NORM

Let M be a smooth Riemannian manifold and let $z \in H_2(M, \mathbb{Z})/\overline{\pi_2(M)}$ where $\overline{\pi_2(M)}$ is the image of the Hurewicz map. In [15], Thurston introduced a seminorm $\|z\|_g$, which we will take in a form

$$\|z\|_g = \inf_{[\Sigma]=z} |\chi(\Sigma)|$$

taken over all singular surfaces $f : \Sigma \rightarrow M$ of genus ≥ 1 , representing z . One makes $\|z\|_g$ to a norm by the standard normalization procedure (see [15]) and shows that this norm is essentially equivalent to the Gromov's simplicial norm.

On the other hand, given a metric on M , one has the usual area norm, or the mass, of z :

$$\|z\|_a = \inf_{[\Sigma]=z} \text{area}(\Sigma).$$

The celebrated Thurston's inequality relates $\|z\|_g$ to $\|z\|_a$ in the case when M is of negative curvature. The following sharp version was established in [7]: if $-K \leq K(M) \leq -k < 0$ and M is compact, then

$$(2) \quad \|z\|_a \leq \frac{2\pi}{k} \|z\|_g.$$

The main purpose of this section is to establish the following result, which is in a sense converse to (2):

THEOREM 1. *Let M be compact three-dimensional homologically atoroidal Riemannian manifold, and let $R(M) = \sup_M(-R(x))$. Then for any $z \in H_2(M, \mathbb{Z})/\overline{\pi_2(M)}$, one has*

$$(3) \quad \|z\|_a \geq \frac{2\pi}{R(M)} \|z\|_g$$

which is sharp.

PROOF: We recall that a three-manifold M is called homologically atoroidal if any map of a torus T^2 to M induces zero homomorphism in the second homology. All hyperbolizable manifolds are atoroidal.

We begin with finding a minimal map $f : \Sigma^g \rightarrow M$ representing z which exists by the Theorem A. We assume $|\chi(\Sigma)|$ is minimal possible and f is an immersion, (Σ^g may be disconnected). Let Σ^{g_i} , $i = 1, \dots, q$, be the components of Σ . Then $g_i > 1$ since $\chi(\Sigma)$ is minimal and M is atoroidal. Thus the second variation formula gives (comp. [13], (5.3))

$$\sum_i \int_{\Sigma^{g_i}} R(x) d\text{area} \leq \sum_i \int_{\Sigma^{g_i}} K d\text{area}$$

which implies by Gauss-Bonnet

$$R(M) \text{Area}(f) \geq 2\pi \sum -\chi(\Sigma^{g_i}) = 2\pi \|z\|_g.$$

Since f is the minimizing map, we get $\text{Area}(f) = \|z\|_a$, so

$$R(M) \|z\|_a \geq 2\pi \|z\|_g,$$

as prescribed by (3).

COROLLARY 2. *Let $T(k, K)$ denotes a class of compact three-manifolds with negative curvature satisfying $-K \leq K(x) \leq -k < 0$. Then the Thurston norm and the area norm are uniformly equivalent in $T(k, K)$. More precisely, for any $M \in T(k, K)$ and $z \in H_2(M, \mathbb{Z})$, one has*

$$(4) \quad \frac{1}{k} \|z\|_g \geq \frac{1}{2\pi} \|z\|_a \geq \frac{1}{3K} \|z\|_g$$

and the left hand side is sharp.

3. GEOMETRY OF JACOBIANS AND CONFORMAL INVARIANTS

For a compact Riemannian manifold N we denote $J^1(N) = H^1(N, \mathbb{R})/H^1(N, \mathbb{Z})$. Let $\Omega^*(N)$ stand for the de Rham complex of N . For $p \geq 1$ and $\omega \in \Omega^k(N)$ we denote as usual $\|\omega\|_{L^p} = (\int_N |\omega|_x^p d\text{Vol})^{1/p}$. This induces a norm on $H^k(N, \mathbb{R})$ by the formula $\|w\|_{L^p} = \inf_{\omega \in w} \|\omega\|_{L^p}$. We claim:

PROPOSITION 3. *Let M be a homologically atoroidal three-manifold with $\pi_2(M) = 0$ and let $0 \neq w \in H_1(M, \mathbb{Z})$. Then*

$$(5) \quad \|w\|_{L^1} \geq \frac{4\pi}{R(M)}.$$

COROLLARY 4. *The volume of the Jacobian $J^1(M)$ in the L^2 -metric is at least $\left(\frac{4\pi}{R(M)\text{Vol}^{1/2}(M)}\right)^m \text{Vol} B_m$, where $m = b_1(M)$, and B_m stands for the Euclidean ball.*

PROOF OF THE PROPOSITION 3: Let $\omega \in \Omega^1(M)$ with $[\omega] = w$. Since all periods of ω are integers, there exists a smooth map $\varphi : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ with $\varphi^*(dt) = \omega$. Let $S(t) = \varphi^{-1}(t)$ and write

$$\int_M \|\nabla \varphi\| d\text{Vol} = \int_0^1 \text{Area}(S(t)) dt$$

by the coarea formula. For almost all t , $S(t)$ is smooth and $[S(t)] \in H_2(M, \mathbb{Z})$ is Poincaré dual to w . Applying Theorem 1, we get

$$\|\omega\|_{L^1} \geq 2\pi R^{-1}(M) \|PD(w)\|_g \geq \frac{4\pi}{R(M)},$$

$$\text{so } \|w\|_{L^1} = \inf_{[\omega]=w} \|\omega\|_{L^1} \geq \frac{4\pi}{R(M)}.$$

Q.E.D.

PROOF OF THE COROLLARY 4: This follows readily from (5) and the Hölder inequality. Let M, w be as in the proposition 3, and let $\omega \in \Omega^1(M)$ with $[\omega] = w$. Let g be the metric of M and let h be its conformal perturbation. Say $h = \varphi \cdot g$ for some positive $\varphi \in C^\infty(M)$. Using the proposition 3, we get

$$R_h(M) \cdot \int_M \|\omega\|_h d\text{Vol}_h \geq 4\pi,$$

or

$$R_h(M) \cdot \int_M \|\omega\|_g \cdot \varphi^2 d\text{Vol}_g \geq 4\pi.$$

This gives

$$\int_M \varphi^2 d\text{Vol}_g \geq \frac{4\pi}{R_h(M) \cdot \|\omega\|_{L_g^\infty}},$$

and, by Hölder,

$$\text{Vol}_h(M) = \int_M \varphi^3 d\text{Vol}_g \geq \left(\frac{4\pi}{R_h(M) \cdot \|\omega\|_{L_g^\infty}} \right)^{3/2} \cdot \text{Vol}_g^{-1/2}(M),$$

so

$$(6) \quad R_h^{3/2}(M) \cdot \text{Vol}_h(M) \geq \left(\frac{4\pi}{\|\omega\|_{L_g^\infty}} \right)^{3/2} \text{Vol}_g^{-1/2}(M).$$

We wish to improve this, letting the original metric g to change within its conformal class.

Put $\hat{g} = \psi \cdot g$ and write (5) for \hat{g} instead of g to get

$$(4\pi)^{3/2} R_h^{-3/2}(M) \text{Vol}_h^{-1}(M) \leq (\sup \|\omega(x)\| \cdot \psi^{-1}(x))^{3/2} \left(\int \psi^3 d\text{Vol}_g \right)^{1/2}.$$

The infimum of the right hand side taken over all $\psi > 0$ is easily seen to be $\|\omega\|_{L_g^3}^{3/2}$, so we get finally

$$4\pi R_h^{-1}(M) \text{Vol}_h^{-2/3}(M) \leq \|\omega\|_{L_g^3}.$$

Letting h be the Yamabe metric in C_g , we arrive to the following result

PROPOSITION 5. *Let M be compact three-dimensional homologically atoroidal manifold with $\pi_2(M) = 0$ and infinite $H_1(M, \mathbb{Z})$. For any conformal class C we have*

$$(7) \quad \text{Vol}(C) \geq \frac{4\pi}{\|w\|_{L^3}},$$

where w is any class in $H^1(M, \mathbb{Z})$ and the L^3 -norm is taken according to any metric $g \in C$.

Observe that $\|w\|_{L_g^3}$ does not depend on the choice of the metric. In fact, L^3 geometry of the Jacobian $J^1(M)$ depends only on C . The number $\inf_{w \in H^1(M, \mathbb{Z})} \|w\|_{L^3}$ denoted $j(C)$, is therefore a conformal invariant. We can write (6) in the form

$$(8) \quad \text{Vol}(C) \geq 4\pi j^{-1}(C).$$

One can view (8) as a three-dimensional version of the Li-Yau estimates, c.f. [5].

THEOREM 2. *Let M be as in the Proposition 5. Then*

$$\sup_C \text{Vol}(C) = \infty.$$

PROOF: Fix $w \in H^1(M, \mathbb{Z})$ and $\omega \in w$. Fix $\varepsilon > 0$. In view of the proposition 5, it is enough to find a metric g on M such that $\|\omega\|_{L_g^3} < \varepsilon$. For that, fix a smooth measure μ on M . We will always assume that the density $\mu|_g = 1$. Set $\|\omega\|_g < \varepsilon^{1/3}$ everywhere and correct g_x if necessary in the kernel of ω_x to achieve $\mu|_g = 1$, keeping $\|\omega\|_g$ unchanged. This would be a desired metric.

REMARK.

We show in [9] that the “most” of homology three-spheres are Haken. It would be very interesting to know if the Theorem 2 is still valid for such manifolds.

4. OBSTRUCTIONS TO NASH EMBEDDINGS

Let M^m and Q^q be Riemannian manifolds with $q \geq \frac{m(m+1)}{2}$. Then any distance-decreasing map $\varphi : M \rightarrow Q$ can be C^0 -approximated by an isometrical embedding (c.f. Gromov [2] for the contemporary survey of related results). In particular, there always exists such an embedding, homotopic to a constant map. The situation changes if we wish to prescribe the topological properties of the embedding, e.x. its action in homology/homotopy groups. For example, if we demand that $\varphi_* : H_1(M) \rightarrow H_1(Q)$ is nonzero, then, evidently, there is a necessary condition that the length of the shortest geodesic in the homology class $C \in H_1(M, \mathbb{Z})$ is not less than that of $\varphi_* C$. Using the machinery developed above, we arrive to more obstructions of global character.

THEOREM 3. *Let M^3 and Q^q be compact Riemannian manifolds. Suppose M is homologically atoroidal with $\pi_2(M) = 0$ and $b_1(M), b_1(Q) \neq 0$. Then there is a constant $C(Q)$, such that if there exists an isometrical immersion $f : M \rightarrow Q$ inducing a nontrivial map in the first real homology, then*

$$R(M) \cdot \text{Vol}(M) \geq C(Q).$$

PROOF: Let $\omega_1, \dots, \omega_r$ be a basis of harmonic 1-forms representing integer classes in $H^1(Q, \mathbb{R})$. Put $C^{-1}(Q) = \max_i \|\omega_i\|_{L^\infty}$. If the action of $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(Q, \mathbb{R})$ is nontrivial, then $[f^*\omega_i] \neq 0$ for some i . Since $[\omega_i] \in H_1(Q, \mathbb{Z})$, also $[f^*\omega_i] \in H_1(M, \mathbb{Z})$. Applying the proposition 3, we get

$$\|f^*\omega_i\|_{L^1} \geq \frac{4\pi}{R(M)}.$$

But $\|f^*\omega_i\|_{L^1} = \int_M \|f^*\omega_i\| d\text{Vol} \leq \|\omega_i\|_{L^\infty} \cdot \text{Vol}(M)$, so $R(M) \cdot \text{Vol}(M) \geq C(Q)$. Q.E.D.

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